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## The gluon splitting function at moderately small $x$

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### Abstract

It is widely believed that at small  $x$ , the BFKL resummed gluon splitting function should grow as a power of  $1/x$ . But in several recent calculations it has been found to *decrease* for moderately small- $x$  before eventually rising. We show that this ‘dip’ structure is a rigorous feature of the  $P_{gg}$  splitting function for sufficiently small  $\alpha_s$ , the minimum occurring formally at  $\log 1/x \sim 1/\sqrt{\alpha_s}$ . We calculate the properties of the dip, including corrections of relative order  $\sqrt{\alpha_s}$ , and discuss how this expansion in powers of  $\sqrt{\alpha_s}$ , which is poorly convergent, can be qualitatively matched to the fully resummed result of a recent calculation, for realistic values of  $\alpha_s$ . Finally, we note that the dip position, as a function of  $\alpha_s$ , provides a lower bound in  $x$  below which the NNLO fixed-order expansion of the splitting function breaks down and the resummation of small- $x$  terms is mandatory.

# 1 Introduction

A major effort is currently under way to push the precision of DGLAP [1–3] splitting functions to next-next-to-leading order (NNLO) accuracy [4]. One of the main applications of such an effort could be to improve the description of the small- $x$  parton distributions, which with the current NLO evolution suffer from pathologies such as negative gluon distributions and predictions of a negative  $F_L$  [5, 6]. Furthermore a good knowledge of small- $x$  parton distributions will be ever-more relevant as collider energies are increased, for example at the LHC or a possible VLHC, which will be able to probe small- $x$  kinematic regions unexplored even at HERA.

However a question that remains to be understood is that of the domain in which fixed order expansions are sufficiently convergent as to be reliable. Indeed it is known that at small  $x$ , there are large logarithmic enhancements of the splitting function at all orders [7, 8], leading formally to the breakdown of the convergence of the series for  $\alpha_s \log 1/x \sim 1$  and it has been argued [9] that there is evidence in the data [10] for the presence of some such terms.

Much effort has been devoted in recent years to resumming these logarithmically enhanced terms, which are expected to lead to a rise at small  $x$ , as a power of  $x$ , for the gluon-gluon splitting function,  $xP_{gg}(x)$ . It turns out however that the LL $x$  summation,  $\alpha_s^n \log^{n-1} 1/x$  rises much too steeply [11, 12] to be compatible with the more gentle rise of the  $F_2$  data [10]. On the other hand, the inclusion of the NLL $x$  terms  $\alpha_s^n \log^{n-2} 1/x$  — extracted from the NLL $x$  kernel eigenvalue [13, 14] and based on several Regge-gluon vertices [15] and on the  $q\bar{q}$  cluster [16, 17] — leads at moderately small  $x$  to a negative splitting function [18, 19]. Since that discovery, there has been investigation of the origin of these problems, and various approaches have been proposed to estimate yet higher orders [20–30], the most successful of them being based on a simultaneous treatment of small- $x$  and collinear logarithms.

A surprising observation, common to all these approaches, is that in the phenomenologically relevant, moderately small- $x$  region, the splitting function actually *decreases*, while the power-like rise is delayed to somewhat smaller values of  $x$  (resummed curve of figure 1 [23], which has been found to be rather close to a splitting function fitted to the  $F_2$  data [27, 30]). The question arises therefore of whether the resulting ‘dip’ structure is a well-defined property of the gluon splitting function, or instead perhaps an artefact of the particular schemes used to ‘improve’ the small- $x$  hierarchy. The purpose of this letter is to show that the dip has a simple origin, specifically in the structure of the first few terms of the perturbative series, possibly matched to a resummed behaviour at smaller  $x$  values.

More precisely (Sec. 2), in the formal limit of small  $\alpha_s$ , the dip is a consequence of an interplay between different fixed orders, and one finds that the simple fixed-order hierarchy breaks down not for  $\alpha_s \log 1/x \sim 1$  as widely expected, but rather for  $\alpha_s \log^2 1/x \sim 1$ . The result is that the properties of the dip can be described in terms of a series in powers of  $\sqrt{\alpha_s}$ . For phenomenologically relevant values of  $\alpha_s$  though, this series in  $\sqrt{\alpha_s}$  turns out to be very poorly convergent. Instead we find that quite simple resummation arguments, presented in section 3, still enable us to gain some quantitative understanding of the dip properties.

## 2 Low perturbative orders and $\sqrt{\alpha_s}$ -expansion

Let us start by recalling the structure of the LL $x$  terms of the  $xP_{gg}(x)$  splitting function,

$$A_{n,n-1} \bar{\alpha}_s^n \log^{n-1} \frac{1}{x}, \quad (n \geq 1), \quad (1)$$

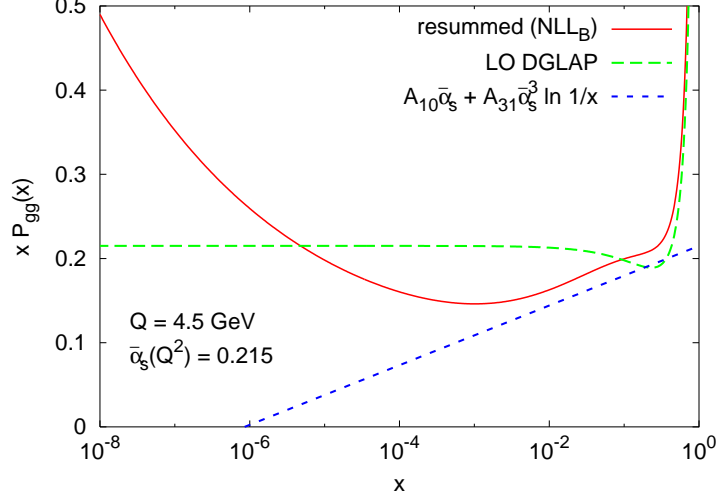


Figure 1: The  $xP_{gg}(x)$  splitting function. The resummed (NLL<sub>B</sub>) curve corresponds to scheme B of [23].

where  $\bar{\alpha}_s = \alpha_s N_c / \pi$ . A number of the lower order terms in the series are absent,  $A_{21} = A_{32} = A_{54} = 0$ , while

$$A_{10} = 1, \quad A_{43} = \frac{\zeta(3)}{3}, \quad A_{65} = \frac{\zeta(5)}{60}, \quad \dots \quad (2)$$

Since these and all further terms are positive, the LL $x$  splitting function grows monotonically as  $x$  decreases. The NLL $x$  terms can be written as

$$A_{n,n-2} \bar{\alpha}_s^n \log^{n-2} \frac{1}{x}, \quad (n \geq 2), \quad (3)$$

where the first few coefficients are [13, 14]

$$A_{20} = -\frac{n_f}{6N_c} \left( \frac{5}{3} + \frac{13}{6N_c^2} \right), \quad (4a)$$

$$A_{31} = -\frac{395}{108} + \frac{\zeta(3)}{2} + \frac{11\pi^2}{72} - \frac{n_f}{4N_c^3} \left( \frac{71}{27} - \frac{\pi^2}{9} \right) \simeq -1.548 - 0.014n_f, \quad (4b)$$

$$A_{42} = -4.054 - 6.010b - 0.030n_f = -9.563 + 0.303n_f, \quad \dots \quad (4c)$$

and  $b = \frac{11}{12} - \frac{n_f}{6N_c}$  is the first beta-function coefficient. These coefficients are given in the  $Q_0$  scheme [31] and for renormalisation scale  $\mu = Q$ . They come from a simple expansion of the NLL $x$  kernel eigenvalue, and — notably  $A_{31}$  and  $A_{42}$  — can be traced back to early calculations of NLL $x$  gluon vertices [15] and of the  $q\bar{q}$  cluster [16, 17]. They include — in particular  $A_{42}$  — the running coupling effects, which are part of the NLL $x$  corrections. In the  $\overline{\text{MS}}$  scheme only the  $n_f$  parts of  $A_{20}$  and  $A_{31}$  will differ, while from  $A_{42}$  onward the  $n_f$  independent part will differ as well. Because of the zeroes in the LL coefficients,  $A_{31}$  and  $A_{42}$  are independent of the choice of  $\mu$ .

The resummation hierarchy as written above in terms of LL $x$  and NLL $x$  terms is intended to be applied when  $\alpha_s \log 1/x$  is of order 1, while  $\alpha_s \ll 1$  and  $\log 1/x \gg 1$ . Let us however examine an intermediate small- $x$  limit in which  $\log 1/x \gg 1$  but  $\alpha_s \log 1/x \ll 1$  (the precise region will be better specified shortly).

Because the LL $x$  coefficients  $A_{21}$  and  $A_{32}$  are zero, the lowest order term with  $\log 1/x$  enhancement is the NLL $x$  term  $A_{31}\alpha_s^3 \log 1/x$ , which is NNLO in the usual DGLAP perturbative expansion. Since  $A_{31}$  is negative it will lead to an initial *decrease* of the splitting function and at some sufficiently small value of  $x$  the NNLO gluon splitting function [32] will become negative, as shown in figure 1, where we have included the small- $x$  part of the NNLO  $xP_{gg}(x)$ ,  $A_{10}\bar{\alpha}_s + A_{31}\bar{\alpha}_s^3 \log 1/x$  ( $A_{20} = 0$  in the particular scheme used in the figure [23]).

At N<sup>3</sup>LO, order  $\alpha_s^4$ , both LL $x$  and NLL $x$  terms are present. Since we are in the regime of  $\log 1/x \gg 1$ , the LL $x$   $\alpha_s^4 \log^3 1/x$  term will clearly dominate over the NLL $x$   $\alpha_s^4 \log^2 1/x$  term. What is interesting however is the interplay between the negative NLL $x$   $\alpha_s^3 \log 1/x$  term and the positive LL $x$   $\alpha_s^4 \log^3 1/x$ :

$$xP_{gg}(x) = \text{const.} + A_{31} \bar{\alpha}_s^3 \log \frac{1}{x} + A_{43} \bar{\alpha}_s^4 \log^3 \frac{1}{x} + \dots, \quad (5)$$

where the constant term includes  $A_{10}\bar{\alpha}_s$  and  $A_{20}\bar{\alpha}_s^2$  contributions, and at each order in  $\alpha_s$  we have written only the term with the strongest  $\log 1/x$  dependence. Since  $A_{43}$  is positive and has stronger  $\log x$  dependence than the negative  $A_{31}$  term, the splitting function as written in (5) will eventually start rising. The  $A_{31}$  and  $A_{43}$  terms will be of the same order when  $\alpha_s \log^2 1/x \sim 1$ , and the splitting function of eq. (5) will have a minimum at

$$\log \frac{1}{x_{\min}} = \sqrt{-\frac{A_{31}}{3A_{43}} \frac{1}{\bar{\alpha}_s}}. \quad (6)$$

The appearance of this minimum for  $\alpha_s \log^2 1/x \sim 1$  suggests that it may be of use to examine an alternative classification of the series, in which we consider all terms that are of similar magnitude when  $\alpha_s \log^2 1/x$  is of order one,<sup>1</sup>

$$A_{k,2k-5} \bar{\alpha}_s^k \log^{2k-5} \frac{1}{x}, \quad (3 \leq k \leq 4). \quad (7)$$

One finds that there are only terms with  $k = 3, 4$ , since lower values of  $k$  would be associated with negative powers of  $\log 1/x$ , while higher values of  $k$  would be super-leading in the usual LL $x$  classification. In other words the two terms,  $\alpha_s^3 \log 1/x$  and  $\alpha_s^4 \log^3 1/x$ , that we have examined so far provide the full leading contribution for  $\alpha_s \log^2 1/x \sim 1$ .

This is illustrated in figure 2, which shows various possible classifications of logarithmically enhanced terms. Rows correspond to a given power of  $\alpha_s$ ; columns to a given single-logarithmic order (LL $x$ , NLL $x$ , and so on); terms on a same downward going diagonal line (reading from left to right) all have the same power of  $\log x$ .

Terms on upward going diagonal lines in figure 2 are of the same order for  $\alpha_s \log^2 1/x \sim 1$ . On any given such diagonal, the number of terms is always finite, due to the fact that the natural hierarchy is single logarithmic. The leading terms in this regime, discussed above in eq. (7), are highlighted by the upper (upward-going diagonal) ellipse. The lower ellipse contains terms suppressed insofar as they have one less power of  $\log 1/x$ ,

$$A_{k,2k-6} \bar{\alpha}_s^k \log^{2k-6} \frac{1}{x}, \quad (3 \leq k \leq 5). \quad (8)$$

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<sup>1</sup>This is a double-logarithmic classification, however one should bear in mind that the perturbative series itself contains at most single logarithms — our study of powers of  $\alpha_s \log^2 1/x$  therefore just represents a particular way of reclassifying terms in the single-logarithmic perturbative expansion.

	LLx	NLLx	NNLLx	...
$\alpha_s$	x	—	—	
$\alpha_s^2$	0	$n_f$	—	
$\alpha_s^3$	0	x	x	
$\alpha_s^4$	x	x	x	$\ln^0 1/x$
$\alpha_s^5$	0	x	x	$\ln 1/x$
$\vdots$				$\ln^2 1/x$ $\ln^3 1/x$

Figure 2: Representation of different classifications of logarithmically enhanced terms. Symbols 'x' indicate terms that are present; '0' indicates terms that could have been present but are zero; ' $n_f$ ' indicates a term whose only non-vanishing part is proportional to  $n_f$ ; a dash indicates terms which do not exist by definition. A red ellipse highlights the leading terms for  $\alpha_s \ln^2 1/x \sim 1$ , and a blue ellipse highlights the subleading terms for  $\alpha_s \ln^2 1/x \sim 1$ .

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Equivalently, since we are interested in the region where  $\log 1/x \sim 1/\sqrt{\alpha_s}$ , these terms are suppressed by a power of  $\sqrt{\alpha_s}$ . Adding the terms of the lower ellipse to eq. (5) one obtains

$$xP_{gg}(x) = \text{const.} + A_{31} \bar{\alpha}_s^3 \log \frac{1}{x} + A_{43} \bar{\alpha}_s^4 \log^3 \frac{1}{x} + A_{42} \bar{\alpha}_s^4 \log^2 \frac{1}{x} + \mathcal{O} \left( \bar{\alpha}_s^k \log^{2k-7} \frac{1}{x} \right), \quad (9)$$

where we have exploited the fact that  $A_{54} = 0$  and that the  $A_{30}$  contribution can be absorbed into the constant piece. Solving for the minimum of eq. (9) gives

$$\log \frac{1}{x_{\min}} = \sqrt{-\frac{A_{31}}{3A_{43}\bar{\alpha}_s} + \frac{A_{42}^2}{9A_{43}^2}} - \frac{A_{42}}{3A_{43}} \quad (10)$$

$$= \sqrt{-\frac{A_{31}}{3A_{43}} \frac{1}{\bar{\alpha}_s}} - \frac{A_{42}}{3A_{43}} + \mathcal{O}(\sqrt{\alpha_s}) \simeq \frac{1.156}{\sqrt{\alpha_s}} + 6.947 + \mathcal{O}(\sqrt{\alpha_s}), \quad (11)$$

where the numerical values have been given for  $n_f = 4$ . We see that the effect of the subleading  $A_{42}$  term is to shift  $\log 1/x_{\min}$  by a (rather large) constant.

As well as considering the position of the dip, it is interesting to study also its depth,  $d$ . Substituting  $\log 1/x \sim \bar{\alpha}_s^{-1/2}$  into eq. (5) one immediately sees that the dip's depth is of order  $\bar{\alpha}_s^{5/2}$ . Including the subleading terms (lower ellipse of figure 2) gives the following result

$$-d = \frac{2A_{31}}{9} \sqrt{\frac{-3A_{31}}{A_{43}}} \bar{\alpha}_s^{5/2} - \frac{1}{3} \frac{A_{31}A_{42}}{A_{43}} \bar{\alpha}_s^3 + \mathcal{O}(\bar{\alpha}_s^{7/2}) \simeq -1.237 \bar{\alpha}_s^{5/2} - 11.15 \bar{\alpha}_s^3 + \mathcal{O}(\bar{\alpha}_s^{7/2}). \quad (12)$$

The depth has been defined with respect to the  $x = 1$  limit of eq. (9), which includes the usual  $\bar{\alpha}_s$  constant term, but also  $A_{20}\bar{\alpha}_s^2$  term and the unknown NNLLx term  $A_{30}\bar{\alpha}_s^3$ . The full  $P_{gg}$  splitting function has of course a  $1/(1-x)_+$  divergence so its  $x = 1$  value can not actually be used as a reference point for defining the depth. So one may choose to define it alternatively with respect to the value of the  $x \rightarrow 0$  LO splitting function,  $A_{10}\bar{\alpha}_s$ . This introduces extra terms  $A_{20}\bar{\alpha}_s^2 + A_{30}\bar{\alpha}_s^3$  in the expression, eq. (12), for  $-d$ .

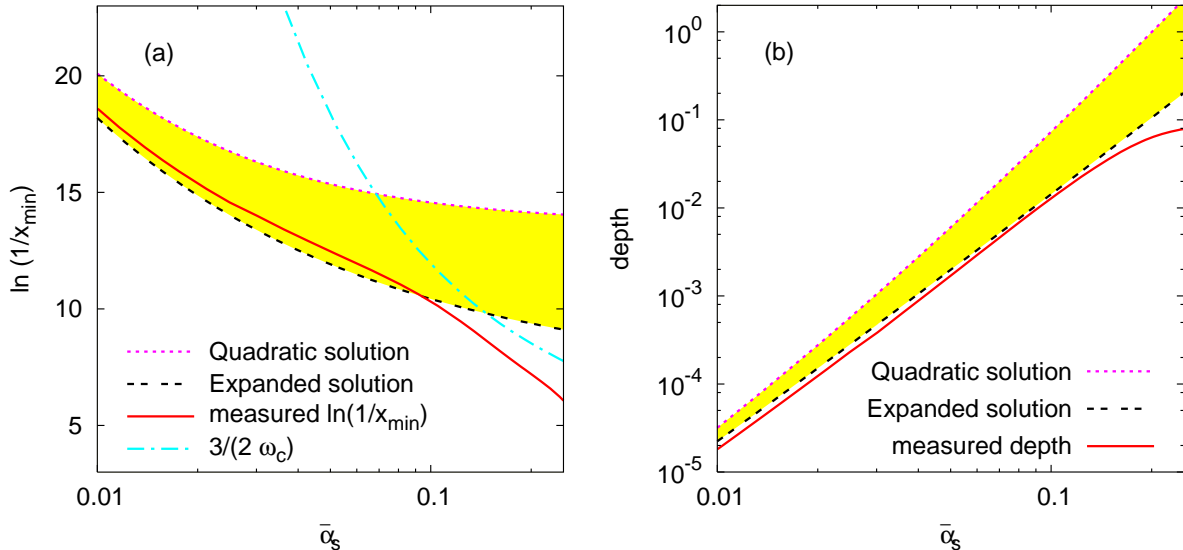


Figure 3: Properties of the dip in the NLL<sub>B</sub> model of [23] compared to our analytical predictions. See text for details.

The dip position and depth, as a function of  $\bar{\alpha}_s$ , are shown respectively in figures 3a and 3b. In each case the solid line represents the dip properties as ‘measured’ from the NLL<sub>B</sub> scheme<sup>2</sup> of [23], which was shown also in figure 1. The shaded band represents the spread of the predictions based on eqs. (9)–(12). The upper edge of the bands, labelled ‘Quadratic solution’ corresponds to the use of eq. (10) and its direct substitution into eq. (9); the lower edge corresponds to eqs. (11) and (12). For small values of  $\alpha_s$ , there is rather good agreement between the expanded forms of our predictions and the dip properties as measured from the full resummation: the dip position is within the uncertainty band, typically close to the expanded solution; the depth is just outside the uncertainty band (again closer to the expanded solution), though this may be because we have measured the depth with respect to the  $A_{10}\alpha_s$  reference level and have not included the resulting additional unknown NNLL $x$   $A_{30}$  contribution to the depth. Instead, including the  $A_{30}$  as it appears in the NLL<sub>B</sub> model, lowers the band so that it overlaps with the measured depth. Leaving aside these details, for both the position and depth of the dip, the scaling with  $\alpha_s$  is clearly reproduced, providing strong evidence that the dip truly is a consequence of the low-order behaviour of the perturbation series.

We note though that the spread of predictions, based on eqs. (10) and (11), is quite significant. This is essentially due to the large value of the  $A_{42}$  coefficient, which means that the series in  $\sqrt{\alpha_s}$  in eqs. (11) and (12) is very poorly convergent — the leading and subleading corrections are of the same order when  $\alpha_s \sim 0.01$ – $0.02$ .

In practice our low-order arguments seem to extend somewhat further, providing a reasonable description of the dip, within the large uncertainties, up to  $\alpha_s \sim 0.05$ – $0.1$ . However

<sup>2</sup>We note that since the NLL<sub>B</sub> scheme accounts only partially for the  $n_f$  dependence (that associated with running of the coupling), the resulting NLL $x$   $A_{n,n-2}$  coefficients differ slightly from those shown in eq. (4), with  $A_{20}$ ,  $A_{31}$  and  $A_{42}$  corresponding to the  $n_f = 0$  results of eq. (4) (in  $A_{42}$  the  $n_f$ -part in the  $b$ -dependent term is retained, hence the coefficient of  $n_f$  in  $A_{42}$  is 0.334). The reason for the only partial inclusion of the  $n_f$  dependence is that the NLL<sub>B</sub> scheme is based on a single-channel, purely gluonic approach, whereas full account of  $n_f$  dependence would require a two-channel, quark-gluon formulation.

beyond this point the prediction fails quite dramatically, with the height of the predicted dip minimum becoming for example negative ( $A_{10}\bar{\alpha}_s - d < 0$ ), in contradiction with the full resummed results. Furthermore there is a clear change in the  $\alpha_s$  dependence for both the measured position and depth of the dip. This suggests that for  $\alpha_s \gtrsim 0.05$  the dip description can no-longer be founded on low-order perturbation theory alone.

### 3 Resummation and cut-representation argument

On the other hand we know that when  $\alpha_s$  is moderate and  $\log 1/x$  is sizeable we enter the usual regime of resummation of terms  $(\alpha_s \log 1/x)^n$  [7], together with its subleading corrections [13, 21]. Though the strict LL $x$ , NLL $x$  hierarchy is ill-behaved, the inclusion of renormalisation group effects tends to stabilise this hierarchy (e.g. [23, 29]). As a result one obtains the usual, expected behaviour of a splitting function that increases as a power of  $x$  at small  $x$ .

A simple estimate of the  $x$  value for which this increase occurs can be obtained in the approximation of a frozen coupling using the quadratic expansion of the effective BFKL characteristic function

$$\bar{\alpha}_s \chi_{\text{eff}}(\gamma, \bar{\alpha}_s) = \omega_s(\bar{\alpha}_s)(1 + D(\bar{\alpha}_s)(\gamma - \gamma_m)^2), \quad (13)$$

where  $\omega_s(\alpha_s)$  is the value of  $\alpha_s \chi$  at its minimum,  $\gamma = \gamma_m$ , and  $D(\alpha_s)$  is related to the second derivative of  $\chi$  (see figures 1 and 3 of [23]).

This leads to the well-known square-root branch-point for the anomalous dimension,

$$\gamma = \gamma_m + \sqrt{\frac{\omega - \omega_s}{D\omega_s}}, \quad (14)$$

and to the representation

$$xP_{gg}(x) \simeq \int^{\omega_s(\bar{\alpha}_s)} \frac{d\omega}{\pi} \sqrt{\frac{\omega_s - \omega}{D\omega_s}} x^{-\omega} \simeq \frac{x^{-\omega_s}}{2\sqrt{\pi\omega_s D} \log^{3/2} 1/x}, \quad (15)$$

for the splitting function.

It is amusing to note that the above estimate shows a dip at

$$\omega_s(\bar{\alpha}_s) \log \frac{1}{x} = \frac{3}{2}, \quad (16)$$

due to the logarithmic prefactor. Of course the actual cut structure of the anomalous dimension is much more complicated, showing a variety of subleading branch cuts, generally at complex  $\omega$  values [11, 18], which are needed to match the small- $x$  representation (15) to perturbation theory for small  $\alpha_s \log 1/x$ . For this reason the dip structure (16), based on the moderate- $x$  behaviour of (15) is not always to be taken seriously.<sup>3</sup>

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<sup>3</sup>For example, as we have mentioned earlier, the fixed-coupling LL $x$  splitting function has no dip at all. It is interesting also to note that LL evolution with (subleading) running coupling corrections does have a dip [26, 33] — its small- $\alpha_s$  properties are different from those of the full NLL $x$  dip, because it is due to an interplay between terms  $\alpha_s^4 \log^n 1/x$  ( $1 \leq n \leq 3$ ) and so, in the limit of small  $\alpha_s$  occurs for  $\log 1/x$  of order 1. A related running-coupling LL $x$  dip has been obtained in [28, 29], though the different scale of the running coupling and the use of the Airy extrapolation mean that it has different formal small- $\bar{\alpha}_s$  properties from [26, 33].

However, in our resummed calculation, the existence of the dip relies on the negative  $\log 1/x$ -slope of the splitting function which is pretty well represented by the  $\sqrt{\alpha_s}$ -expansion, as noticed before. Furthermore, for  $\omega_s \log 1/x \gtrsim 3/2$ , eq. (15) is a reasonable representation of the splitting function and in cases — as ours — in which there is a dip, we can take eq. (16) as an *upper bound* on its position.

In the running-coupling case it is to be kept in mind that the cut is actually broken up into a series of poles, the leading one being at a position  $\omega_c(\alpha_s)$  which lies somewhat below  $\omega_s(\alpha_s)$  (see figure 18 of [23]) because of running coupling effects. Nevertheless, as long as  $x$  is not too small, the inverse Mellin transform (15) does not resolve the difference between a cut and series of poles.

Therefore, by joining the  $\sqrt{\alpha_s}$ -expansion with the cut-representation arguments, we are led to believe that the perturbative and resummed regions can be matched by the inequality

$$\log \frac{1}{x_{\min}} \simeq \frac{c_1}{\sqrt{\bar{\alpha}_s}} + c_2 \lesssim \frac{3}{2\omega_c(\bar{\alpha}_s)}, \quad (17)$$

where  $c_1$  and  $c_2$  are provided by eq. (11), and we have replaced  $\omega_s$  with  $\omega_c \lesssim \omega_s$ . Since the right-hand expression goes as  $1/\bar{\alpha}_s$ , this equation provides a transition point in  $\bar{\alpha}_s$ , below which one should use the perturbative (double-logarithmic) representation described before, and above which one should use the full resummed behaviour.

This is confirmed by the moderate  $\bar{\alpha}_s$  region of figure 3a, where one sees a clear bend in the behaviour of  $\log 1/x_{\min}$  when  $3/2\omega_c$  becomes of the same order as the perturbative representation, eq. (11), with the measured  $\log 1/x_{\min}$  remaining consistently below  $3/2\omega_c$ .

## 4 Conclusions

The arguments provided in this letter go some way towards explaining the features of the dip for a range of  $\alpha_s$  values, both in terms of a perturbative series in powers of  $\sqrt{\alpha_s}$  for small  $\alpha_s$ , and in terms of a resummed upper bound of the dip position,  $\sim 3/(2\omega_c)$ , for moderate values of  $\alpha_s$ .

It is the moderate- $\alpha_s$  region that remains the least well understood, the matching of the small- $x$  increase to the initial decrease being a quite non-trivial problem. For example the simple resummed treatment given above is subject to additional running-coupling effects (e.g. difference between  $\omega_c$  and  $\omega_s$ ) which may contribute further displacement of the dip and which have not been considered here. Nevertheless, the arguments given so far show that the dip does exist, as a moderately small- $x$  phenomenon, under the simple condition that the small- $x$  part of  $xP_{gg}(x)$  has initially a negative  $\log 1/x$  slope, as is the case starting at NNLO.

An important phenomenological point that remains to be made concerns the validity of fixed-order expansions of the splitting functions. From our analysis of the dip properties, it is clear that for  $\alpha_s \gtrsim 0.05$  one starts to see a breakdown of the perturbative expansion. Despite this fact one notes a remarkable property of figure 1, namely that the pure NNLO expansion of the splitting function coincides rather well with the resummed result up to the position of the dip minimum — considerably beyond the point in  $x$  where one would have naively expected the  $\alpha_s^4$  DGLAP terms to completely change the behaviour of the splitting function. This holds for a wide range of  $\alpha_s$ .

We cannot claim to have fully understood this observation, however it does suggest that it may in general be safe to use the fixed order, NNLO,  $P_{gg}(x)$  splitting function down to  $x$  values



corresponding to the dip position, and only beyond this point will small- $x$  resummation be strictly necessary. Thus one can use the ‘measured’ dip position, the solid curve of figure 3a, as an estimate of the limit of validity of the NNLO expansion at small  $x$ . Considering this in the context of the available  $F_2$  data, one sees that the limit cuts through the HERA kinematical range, suggesting that while much of the data will be in the region that is ‘safe’ for an NNLO analysis, there is also a substantial region at lower  $x$  and  $Q^2$  in which resummation will be needed.

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